Solving Laplace's Equation in Rectangular Domains

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May 25, 2010

Let S be the square in $\mathbb{R}^2$ with $0 \le x, y \le \pi$. We wish to find explicit formulas for harmonic functions in S when we only know boundary values. We'll only consider a special case—when the function vanishes (that is, equals zero) on three sides of the square. To be explicit, we want to solve this problem:

> Solve $\Delta u = 0$ inside S with the constraint that $u(\pi, y) = f(y)$ for $0 < y < \pi$ and $u = 0$ on the other three sides of S.

If we wanted to find harmonic functions with more general boundary behavior, we'd only have to add together functions of the above type. That's why our problem is sufficient to consider.

Let's test for solutions. Suppose that $u(x, y) = F(x)G(y)$ for some single-variable functions F, G. Feeding this into Laplace's equation gives (after rearranging)

$$-\frac{F''(x)}{F(x)} = \frac{G''(y)}{G(y)}.$$

Each side of this equation is a function of a different variable; thus each side must be constant. For now, let's call the constant k. This gives two ODEs.

$$F''(x) + kF(x) = 0$$
$$G''(y) - kG(y) = 0$$

We have to examine some cases now. If $k = 0$, both F and G are linear functions. In particular, if u matches those zero boundary conditions, then $F, G \equiv 0$. This case is uninteresting, so we exclude $k = 0$.

Suppose $k > 0$. Then G is a sum of exponentials, and matching the zero boundary conditions requires $G \equiv 0$. This case is also uninteresting, so we exclude $k > 0$. We must have $k < 0$, so let's write $k = -a^2$ for some real number a (which might as well be positive, it makes no difference).

Solving the ODEs gives

$$F(x) = A\cosh(ax) + B\sinh(ax)$$
$$G(y) = C\cos(ay) + D\sin(ay)$$

Since $u(0, y) = 0$, we must have $F(0) = 0$. This gives $A = 0$. Since $u(x, 0) = 0$, $G(0) = 0$. Thus $C = 0$. Finally, since $u(x, \pi) = 0$, we must have $G(\pi) = 0$. Therefore $\sin(a\pi) = 0$; that is, a must be a positive integer.

Altogether, we have found that <u>if</u> u is harmonic, satisfies the three zero boundary conditions, and is of the form $F(x)G(y)$, then it must have the form

$$u_n(x, y) = L_n \sinh(nx)\sin(ny),$$

where L_n is a constant. Note that we have changed a into n, to better reflect that it is an integer.

To find a solution which matches the given fourth boundary condition, $u(\pi, y) = f(y)$, we take infinite sums of the functions u_n.

$$u(x, y) = \sum_{n=1}^{\infty} L_n \sinh(nx)\sin(ny).$$

We pick the constants L_n to match the boundary data. Plugging in $x = \pi$ gives

$$f(y) = u(\pi, y) = \sum_{n=1}^{\infty} L_n \sinh(n\pi)\sin(ny).$$

The function f was only defined on $0 < y < \pi$; if we extend (that is, pretend) the function to be an odd function on $-\pi < y < \pi$, then the above equation is the Fourier series of f! To pick the constants L_n, just define

$$L_n = \frac{b_n}{\sinh(n\pi)} = \frac{2}{\pi\sinh(n\pi)} \int_0^{\pi} f(y)\sin(ny)\,dy.$$